Graph Theory

Lebanese University
Faculty of Science
BS Computer Science
2nd year – Fall Semester

Syllabus

1. Introductory concepts
2. Introduction to graphs and their uses
3. Trees and bipartite graphs
4. Distance and connectivity
5. Matrices
6. Graph algorithms
7. Eulerian and Hamiltonian graphs
8. Graph coloring
9. Planar graphs
10. Digraphs and Networks

Matrices

Week 5

Before we start

- Make sure to do the part ”Review of matrix concepts” by your own before attending the lecture.

Outline

- Review of matrix concepts
- The adjacency matrix
- The distance matrix

Review of matrix concepts

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A **matrix** is a rectangular table of numbers, which is usually denoted using a capital letter.
- The numbers within the matrix are called **entries** of the matrix.
- The horizontal lists of numbers are **rows** of the matrix and the vertical lists are **columns**.

Review of matrix concepts

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The **dimension** of a matrix with m rows and n columns is denoted by $m \times n$ (read "m by n").
- If $m = n$, the matrix is **square**.
- The entry that is in row i and column j is called the (i, j) **−entry**, and its **address** is (i, j). If the matrix is represented by A, we denote its (i, j) −entry by a_{ij}.

Matrix arithmetic

- Matrices A and B are **equal** – that is, $A = B$ provided they have same dimension say $m \times n$, and $a_{ij} = b_{ij}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$. In other words, their corresponding entries are equal.

Matrix arithmetic

- To add two matrices, their dimensions must be the same; that is, they must both be both $m \times n$ matrices. In that case, the **sum** of matrices A and B is a matrix C whose entries satisfy the equation $c_{ij}=a_{ij} + b_{ij}$.
- Their **difference** is obtained similarly. Thus, if we subtract matrix B from matrix A, we get a new matrix D whose entries satisfy $d_{ij}=a_{ij} - b_{ij}$.
- Since addition of real numbers is commutative, it should be obvious from the definition that matrix addition is commutative; that is, $A + B = B + A$. Equally obvious is the fact that, in general, $A - B \neq B - A$.

Matrix arithmetic

- Since multiplication is repeated addition (3*17 = 17+17 +17), it makes sense to define the product of a number r and a matrix M. **Scalar multiplication** consists of a scalar (number) r times a matrix M, which produces the new matrix rM, whose (i, j) −entry is $r * m_{ij}$. In other words, we distribute r to each entry of M.

Matrix arithmetic

- The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix, denoted A^t, obtained from A by interchanging its rows and columns. That is, the first row of A becomes the first column of A^t, the second row of A becomes the second column of A^t, and so on. Similarly, the first column of A becomes the first row of A^t, and so on. This is equivalent to defining A^t as the matrix whose (i, j) −entry is the (j, i) −entry of A.
- Note that a square matrix A for which $A = A^t$, it is called **symmetric**.

Matrix arithmetic

- The **diagonal** of an $n \times n$ matrix A consists of the entries a_{ii}, $1 \leq i \leq n$; that is, the entries whose row and column numbers are identical. The $n \times n$ matrix whose diagonal entries each equal 1 and whose remaining Entries each equal 0 is called the **identity matrix** and is denoted I.
- Note that the diagonal entries remain in place when we find the transpose.

Matrix arithmetic

- A $1\times n$ matrix is called a **row matrix** and an $n\times 1$ matrix is called a **column matrix**. In either of those cases, the word “vector” is sometimes used in place of “matrix.” That is, a vector is a matrix that has a single row or a single column.
- A matrix is a rectangular array of numbers, while a vector is a linear array of numbers. A vector is sometimes thought of as an ordered n-tuple of numbers. The order matters. Of course, the entries of a vector need only one coordinate as an address. Also, a vector is usually named using a lowercase letter. So the entries of an $n\times 1$ vector v would be $v_1, v_2, v_3, \ldots, v_n$. We call v_i the ith coordinate or the ith entry of v.

Matrix arithmetic

- Given an $m \times r$ matrix A and an $r \times n$ matrix B, the **product** $C = AB$ is an $m \times n$ matrix.
- Furthermore, the (i, j) −entry of C, c_{ij}, is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \sum_{k=1}^{r} a_{ik}b_{kj}$$

Matrix arithmetic

- Matrix multiplication is not commutative, i.e. $AB \neq BA$.
- Matrix multiplication is associative, i.e. $(AB)C = A(BC)$.
- We define **powers of a square matrix** as follows: $A^2 = AA$, $A^3 = AAA$, and so on.
- Assume that A is $n \times n$ matrix and I is the $n \times n$ identity matrix. Then
  1. $A^r A^s = A^{r+s}$
  2. $(A^r)^s = A^{rs}$
  3. $IA = AI = A$
- We define A^0 as I.
- Notice also that $A^r A^s = A^s A^r$; that is, powers of A commute.

Review of matrix concepts Question 1-10

For the following review questions, let

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 5 & 2 \\ 6 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 3 & 3 \\ 1 & 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 6 \\ 1 & 4 \\ 0 & 6 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Review of matrix concepts Question 1

Find AB, AC, and BC.

Solution:

$$AB = \begin{bmatrix} 13 & 15 & 13 \\ 2 & 19 & 19 \\ 28 & 17 & 11 \end{bmatrix}, AC = \begin{bmatrix} 5 & 26 & 18 \\ 5 & 15 & 4 \\ 13 & 21 & 32 \end{bmatrix}, BC = \begin{bmatrix} 9 & 1 & 16 \\ 3 & 18 & 6 \\ 4 & 12 & 8 \end{bmatrix}$$

Review of matrix concepts Question 2

Show that $AB \neq BA$.

Solution:

$$AB = \begin{bmatrix} 13 & 15 & 13 \\ 2 & 19 & 19 \\ 28 & 17 & 11 \end{bmatrix}, BA = \begin{bmatrix} 8 & 9 & 22 \\ 18 & 18 & 18 \\ 14 & 13 & 17 \end{bmatrix}$$

Review of matrix concepts
Question 3

Find $(AB)^t$ and show that it equals $B^t A^t$.

Solution:

$$(AB)^t = \begin{bmatrix} 13 & 2 & 28 \\ 15 & 19 & 17 \\ 13 & 19 & 11 \end{bmatrix}$$

Review of matrix concepts
Question 4

Verify that $(AB)C = A(BC)$.

Solution:

$$(AB)C = \begin{bmatrix} 41 & 80 & 78 \\ 23 & 114 & 46 \\ 73 & 72 & 134 \end{bmatrix}$$

Review of matrix concepts
Question 5

Find Av. Why can't we find Dv?

Solution:

$$Av = \begin{bmatrix} 33 \\ 20 \\ 40 \end{bmatrix}$$

D is a $3{\times}2$ matrix and v is a $3{\times}1$, so we can't find Dv.

Review of matrix concepts
Question 6

Show that $(A+B)v = Av + Bv$.

Solution:

$$(A+B)v = \begin{bmatrix} 6 & 2 & 5 \\ 0 & 8 & 5 \\ 7 & 3 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ 41 \\ 57 \end{bmatrix}$$

$$Av + Bv = \begin{bmatrix} 33 \\ 20 \\ 40 \end{bmatrix} + \begin{bmatrix} 14 \\ 21 \\ 17 \end{bmatrix} = \begin{bmatrix} 47 \\ 41 \\ 57 \end{bmatrix}$$

Review of matrix concepts
Question 7

Find v^t.

Solution:

$$v^t = \begin{bmatrix} 3 & 2 & 5 \end{bmatrix}$$

Review of matrix concepts
Question 8

Find $A+B$, $3C-2B$, and $I+A+10B$.

Solution:

$$I + A + 10 = \begin{bmatrix} 43 & 11 & 5 \\ 0 & 36 & 32 \\ 16 & 21 & 25 \end{bmatrix}$$

Review of matrix concepts
Question 9

Which of the following are undefined: DA, $C + D$, CD, $A + v$, D^2 and ID?

Solution:

DA, $C + D$, $A + v$ and D^2

Review of matrix concepts
Question 10

Find $(A+B)^2$. Is this equal to $A^2+2AB+B^2$? If not, how do you explain this?

Solution:

$(A+B)^2 = (A+B)(A+B) = A(A+B)+B(A+B) = A^2+AB+BA+B^2 \neq A^2+AB+AB+B^2$.

This is because $AB \neq BA$.