Lebanese University
Faculty of Science
Computer Science BS Degree

Advanced Algorithms I3341

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Course Chapters

as per the textbook

1. Introduction
2. Data Structures and Libraries
3. Problem Solving Paradigms
4. Graph
5. Mathematics
6. String Processing
7. Computational Geometry

Computational Geometry

Chapter 7

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Typical issues with geometry problems

- ++ tricky corner test cases
  - vertical line → infinite gradient
  - collinear points
  - concave polygon
  - case where convex hull of a set of points is the set of points itself
- → do lots of corner test cases
- floating point precision errors
- tedious coding

Chapter Outline

1. Geometry Basics
   a. Points, Lines
   b. Circles
   c. Triangles
   d. Polygons
2. Algorithms on 2D polygons
   a. convex polygon
   b. concave polygon
   c. point inside | outside polygon
   d. cut polygon with straight line
   e. find convex hull of set of points

Not discussed

1. ~~Quadrilaterals~~
2. ~~3D Objects: Spheres~~
3. ~~Other 3D Objects: Cones, Cylinders, etc.~~
4. ~~Plane Sweep technique~~
5. ~~Intersection problems~~
6. ~~Divide and Conquer in geometry problems~~

Some Comp Geometry Principles

- highlight special cases
  - choose implementation to reduce them
- prefer test (predicates) over computing exact numerical answers
- floating point operations
  - avoid division, square root, …
  - operate in integers
  - floating point equality test: fabs(a - b) < EPS where EPS is a small number like 1e-9 instead of a == b

Geometry Basics

Point Simplest Representation

- point_i → integer
- basic raw form, minimalist mode
- whenever possible, work with point_i

```
class point_i
{
    int x, y;
        point_i() {
                x = y = 0;
    }

        point_i(int _x, int _y) {
                x = _x;
                y = _y;
    }
}
```

Point + Sorting Feature

- only used if more precision is needed
- compareTo
  - use EPS (1e-9) to test float equal
  - useful for sorting
    - first: by x-coordinate
    - second: by y-coordinate

```
class point implements Comparable<point> {
   double x, y;

   point() {
      x = y = 0.0;
   }
   point(double _x, double _y) {
      x = _x;
      y = _y;
   }

   @Override
   public int compareTo(point other) {
      if (Math.abs(x - other.x) > EPS)
         return (int)
           Math.ceil(x - other.x);
      if (Math.abs(y - other.y) > EPS)
         return (int)
            Math.ceil(y - other.y);

      return 0;
   }

}
```


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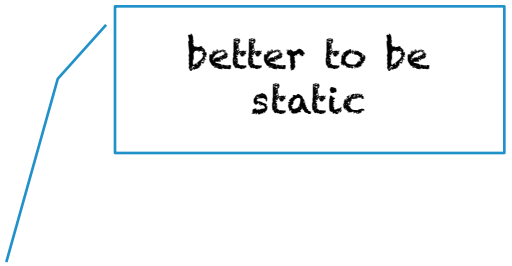
Euclidean Distance between two points

- Math.hypot(dx, dy)
  - returns sqrt(dx * dx + dy * dy)
  - return double

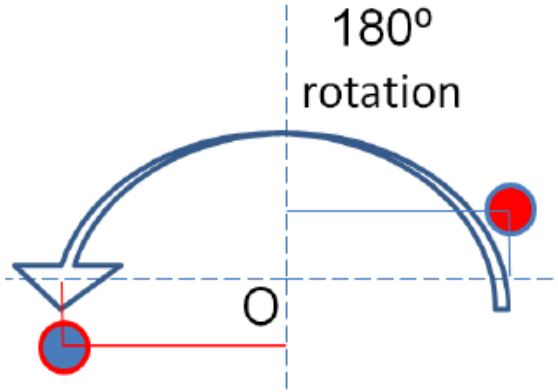



```
static double dist(point p1, point p2)
{
   return Math.hypot(p1.x - p2.x, p1.y - p2.y);
}
```

Rotate a point

- rotate a point
  - by angle θ
- counter clockwise (CCW)
- around O
  - using a rotation matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$



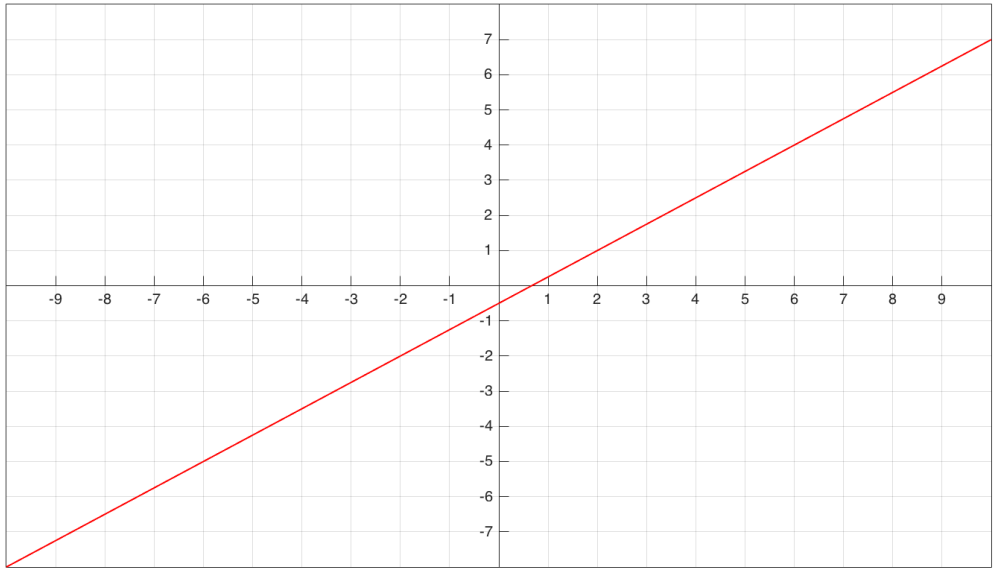

```
double DEG_to_RAD(double d){
  return d * Math.PI / 180.0;
}

double RAD_to_DEG(double r){
  return r * 180.0 / Math.PI;
}

point rotate(point p, double theta){
  double rad = DEG_to_RAD(theta);
  return new point(p.x * Math.cos(rad)
                       - p.y * Math.sin(rad),
                   p.x * Math.sin(rad)
                    + p.y * Math.cos(rad));
}
```

Lines

ch7_01_points_lines.cpp/java

Equation of Straight Line

- any straight line can be written in the form
  - $ax + by + c = 0$
- example
  - $y = \frac{3}{4}x - \frac{1}{2}$
  - can be written as
  - $4y - 3x + 2 = 0$
- in particular
  - equations $x = c$
  - cannot be as form $y = mx + c$



Lines

- poor line equation
  - y = mx + c
  - vertical line → special case


```
class line2 {
      double m, c;
}
```

- better line equation
  - ax + by + c = 0


```
class line {
      double a, b, c;
}
```

pointsToLine

- calculate a, b, c
  - → fill fields of 3rd parameter
- p1(x_1, y_1)
  - $ax_1 + by_1 + c = 0$ (1)
- p2(x_2, y_2)
  - $ax_2 + by_2 + c = 0$ (2)

- (1)-(2)→
  - $a(x_1 - x_2) + b(y_1 - y_2) = 0$
  - if $\partial x = x_1 - x_2 = \varepsilon$
    - vertical line $b = 0, a = 1, c = -x_1$
  - else
    - fix $b = 1.0$
    - $a(x_1 - x_2) + (y_1 - y_2) = 0$
    - $a = -\frac{\partial y}{\partial x}$ and $c = -ax_1 - y_1$

```
void pointsToLine ( point p1, point p2, line l)
{
   if (Math.abs(p1.x - p2.x) < EPS) {
      l.a = 1.0;
      l.b = 0.0;
      l.c = -p1.x;
   }
   else {
      l.a = -(double) (p1.y - p2.y)
            / (p1.x - p2.x);
      l.b = 1.0;
      l.c = -(double) (l.a * p1.x) - p1.y;
   }
}
```

Interaction between two lines

- areParallel
  - check coefficients a & b

```
boolean areParallel(line l1, line l2) {
  return (Math.abs(l1.a-l2.a)<EPS)
          &&(Math.abs(l1.b-l2.b)<EPS);
}
```

- areSame
  - ++ check coefficient c

```
boolean areSame(line l1, line l2) {
  return areParallel(l1, l2)
          &&(Math.abs(l1.c-l2.c)<EPS);
}
```

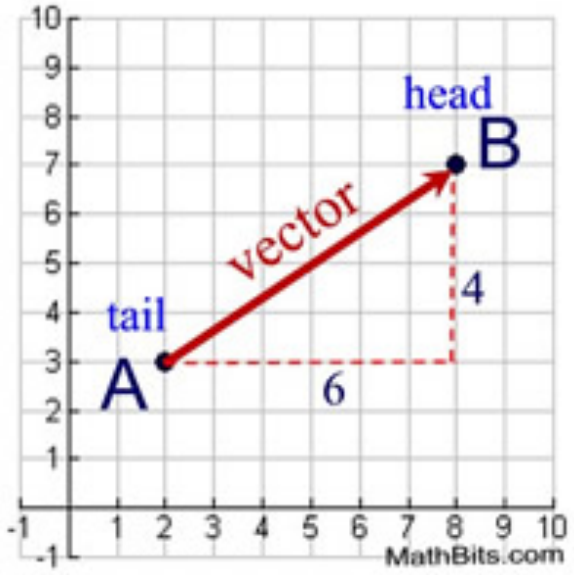
areIntersect

- if parallel → no
- else solve system of 2 linear algebraic equations with 2 unknowns
  - line1: $a_1x + b_1y + c_1 = 0$ (1)
  - line2: $a_2x + b_2y + c_2 = 0$ (2)
- x
  - $b_1 \times (2) - b_2 \times (1) = 0$
  - $x = \frac{b_2c_1 - b_1c_2}{a_2b_1 - a_1b_2}$
- y
  - b either 0 (vertival) or 1
  - avoid division by zero
  - →replace x value in non vertical equation

```
boolean areIntersect(line l1, line l2, point p) {
   if (areParallel(l1, l2))
     return false;
   p.x = (l2.b * l1.c - l1.b * l2.c)
         / (l2.a * l1.b - l1.a * l2.b);

   if (Math.abs(l1.b) > EPS)
      p.y = -(l1.a * p.x + l1.c);
   else
      p.y = -(l2.a * p.x + l2.c);
   return true;
}
```

Segment and Vector

- segment
  - line with two endpoints
  - finite length
- vector
  - segment with direction
- toVec
  - convert 2 points to vector




```
class vec {
  double x, y;
  vec(double _x, double _y) {
    x = _x;
    y = _y;
  }
}

vec toVec(point a, point b) {
  return new vec(b.x - a.x, b.y - a.y);
}
```

Scale and Translate

- scale a vector:
  - nonnegative s = [<1 .. 1 .. >1]
  - shorter .. same .. longer

```
vec scale(vec v, double s) {
  return new vec(v.x * s, v.y * s);
}
```

- translate: move point % vec

```
point translate(point p, vec v) {
  return new point(p.x + v.x, p.y + v.y);
}
```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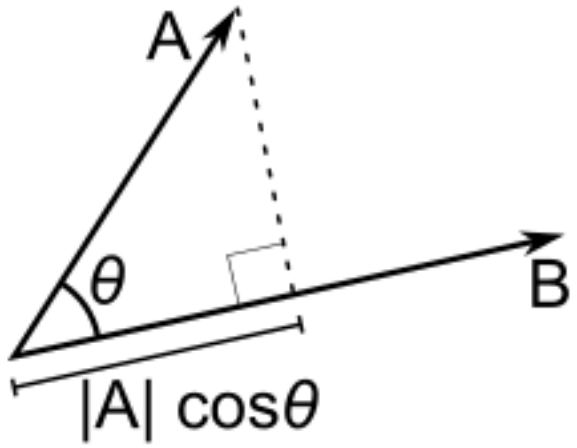
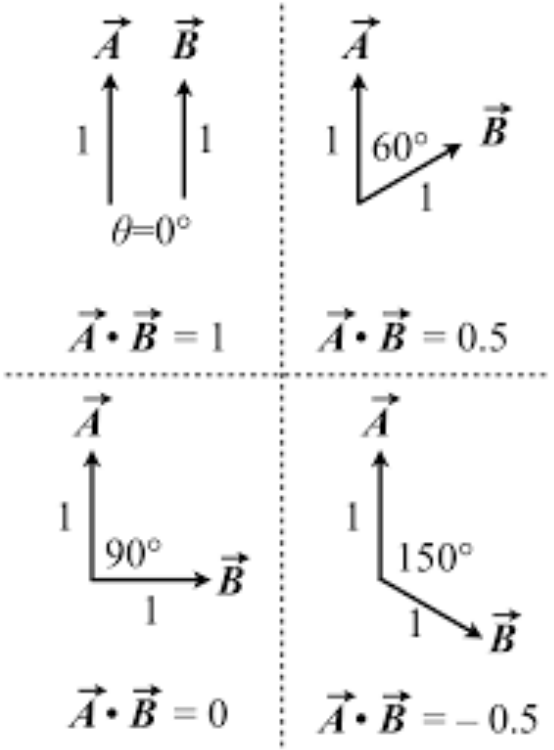
Dot Product

- dot product of 2 vectors
  - u=(a,b) and v=(c,d)
  - is denoted $u \cdot v$ read u dot v

- $\vec{u} \cdot \vec{v} = ac + bd$
  - if >0 → angle acute
  - if =0 → right angle
  - if <0 angle obtuse

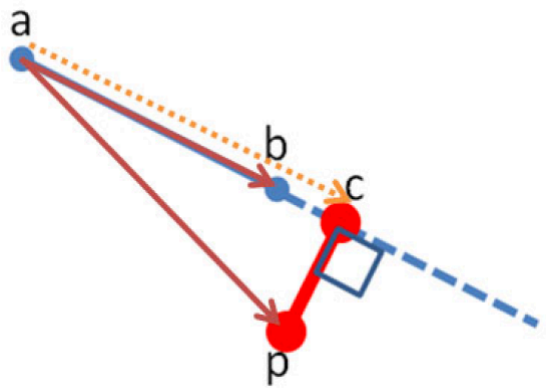



```
double dot(vec a, vec b) {
  return (a.x * b.x + a.y * b.y);
}
```



Distance to Line

- distance from p to line defined by a and b (distinct)
  - closest point stored in 4th param
  - formula: c = a + u * ab
  - translate a to c
  - dist(p,c)

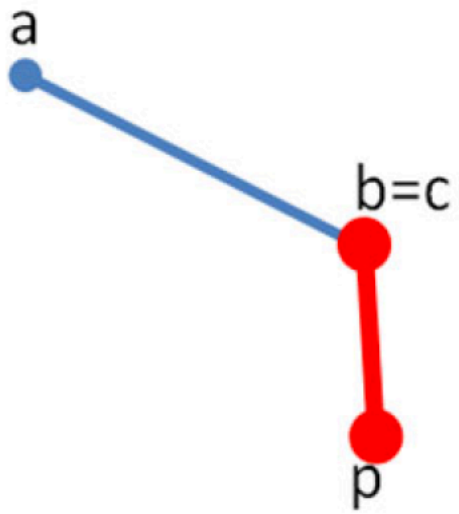



```
double norm_sq(vec v) {
  return v.x * v.x + v.y * v.y;
}
```

```
double distToLine(point p, point a, point b,
                  point c) {
  vec ap = toVec(a, p), ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);
  c = translate(a, scale(ab, u));
  return dist(p, c);
}
```

Distance to Line Segment

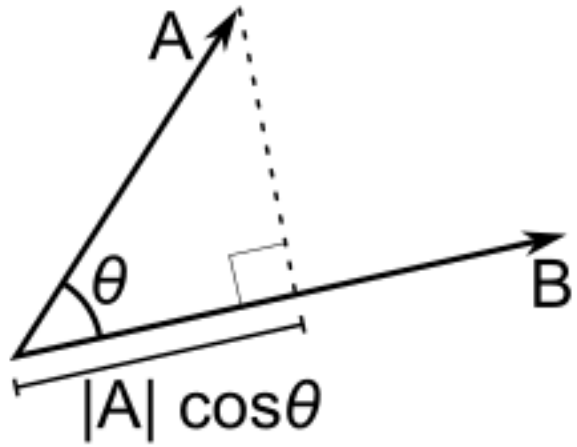
- dist from p to segment ab
- OK if a == b
- closest point stored in 4th param




```
double distToLineSegment(point p, point a,
       point b, point c) {
  vec ap = toVec(a, p),
      ab = toVec(a, b);
  double u = dot(ap, ab) / norm_sq(ab);

  if (u < 0.0) // closer to a
  {
    c = new point(a.x, a.y);
    return dist(p, a);
  }

  if (u > 1.0) // closer to b
  {
    c = new point(b.x, b.y);
    return dist(p, b);
  }

  return distToLine(p, a, b, c);
}
```

Angle

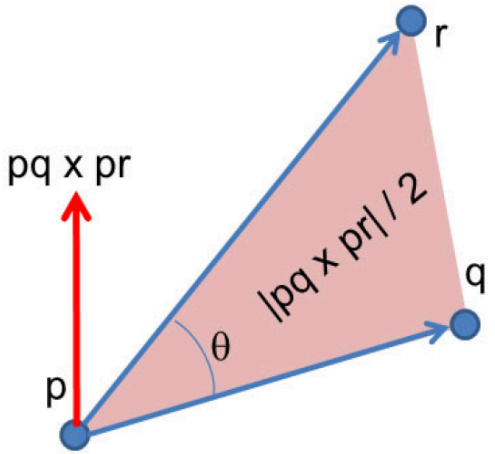
- angle in radiant aob using dot product
  - $oa \cdot ob = |oa||ob|cos(\theta)$
  - $theta = arcos(oa \cdot ob/|oa||ob|)$




```
double angle(point a, point o, point b) {
  vec oa = toVec(o, a),
      ob = toVec(o, b);

  return Math.acos(dot(oa, ob)
         / Math.sqrt(norm_sq(oa)
           * norm_sq(ob)));
}
```

Cross Product, CCW and Collinear Points

- line p q + point r
  - r on left | right side of the line?
  - 3 points p, q, r collinear?
- pq & pr 2 vectors
  - cross product $pq \times pr$ → vector ⊥
  - magnitude = area parallelogram span
    - > 0: p → q → r is a left turn
    - = 0: p, q , r are collinear
    - < 0: p → q → r is a right turn




```
double cross(vec a, vec b) {
  return a.x * b.y - a.y * b.x;
}

boolean ccw(point p, point q, point r) {
  return cross(toVec(p, q),
               toVec(p, r)) > 0;
}

boolean collinear(point p, point q, point r) {
  return Math.abs(cross(toVec(p, q),
                        toVec(p, r))) < EPS;
}
```

Circles

ch7_02_circles.cpp/java

Circle

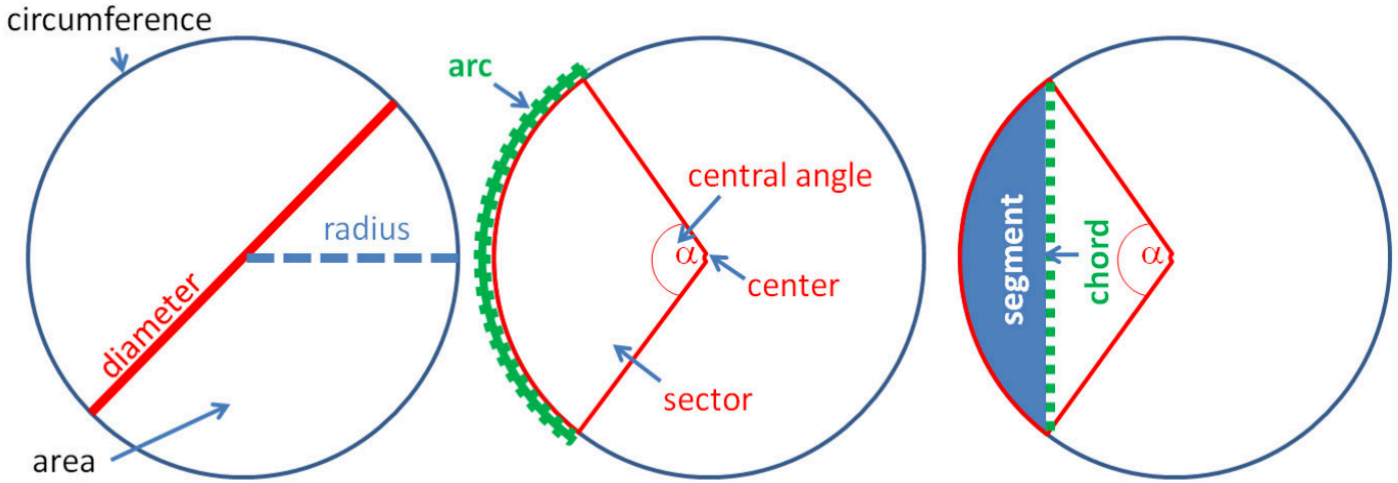
- circle
  - centered at (a, b)
  - radius r
  - set of points (x, y) /
  - $(x-a)^2+(y-b)^2=r^2$
- inCircle
  - 0 – inside
  - 1 – at border
  - 2 – outside
- modify for float

```
int insideCircle(point_i p, point_i c, int r) {
  int dx = p.x - c.x,
      dy = p.y - c.y;
  int Euc = dx * dx + dy * dy,
      rSq = r * r;

  return Euc<rSq? 0:Euc == rSq? 1:2;
}
```

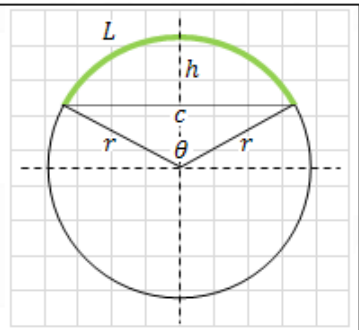
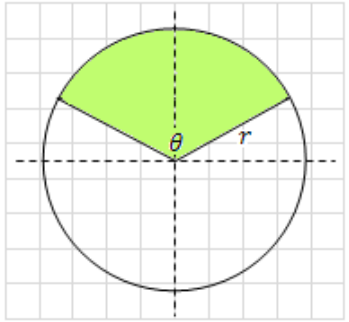
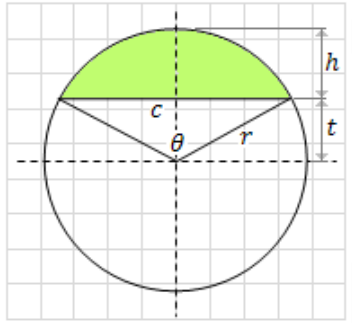
Circle

- for precision use
  - π = 2 * acos(0.0)
- diameter $d = 2r$
- circumference $c = \pi d$
- area $A = \pi r^2$

- $arc = \frac{\alpha}{360.0} \times c$
- $chord = \sqrt{2r^2(1 - cos(\alpha))}$
- $sector = \frac{\alpha}{360.0} \times A$
- segment area = sector –area of isosceles triangle with sides: r, r, chord



Summary

	$L = Arc\ length$ $L(r,\theta) = r\theta$ $L(r,c) = 2r\sin^{-1}\left(\frac{c}{2r}\right)$ $L(r,h) = 2r\cos^{-1}\left(1-\frac{h}{r}\right)$	$L = r\frac{\theta}{180}\pi$
Circle sector 	(θ in radians) $A(r,\theta) = \frac{\theta}{2}r^2$ $A(r,L) = \frac{1}{2}Lr$ $A(r,c) = r^2\sin^{-1}\left(\frac{c}{2r}\right)$	(θ in degree) $A = \frac{\theta}{360}\pi r^2$
	$P(r,\theta) = r(\theta+2)$ $P(r,c) = 2r\left(\sin^{-1}\left(\frac{c}{2r}\right)+1\right)$	$P = r\left(\frac{\theta}{180}\pi+2\right)$
Circle segment 	(θ in radians) $A(r,\theta) = \frac{r^2}{2}(\theta-\sin\theta)$ $A(r,c) = r^2\sin^{-1}\left(\frac{c}{2r}\right)-\frac{c}{4}\sqrt{4r^2-c^2}$ $A(r,t) = r^2\cos^{-1}\left(\frac{t}{r}\right)-t\sqrt{r^2-t^2}$ $A(r,h) = r^2\cos^{-1}\left(1-\frac{h}{r}\right)-(r-h)\sqrt{2rh-h^2}$	(θ in degrees) $A = \frac{r^2}{2}\left(\frac{\pi}{180}\theta-\sin\theta\right)$
	$P(r,\theta) = r\left(\theta+2\sin\frac{\theta}{2}\right)$ $P(r,c) = r^2\sin^{-1}\left(\frac{c}{2r}\right)-\frac{c}{2}\sqrt{4r^2-c^2}$	$P = r\left(\frac{\pi}{180}\theta+2\sin\frac{\theta}{2}\right)$

source http://www.ambrsoft.com/TrigoCalc/Sphere/Arc_.htm

Find the center

- p1 & p2 + radius r
  - → ? centers (c1 & c2)
- to get the other center, reverse p1 and p2

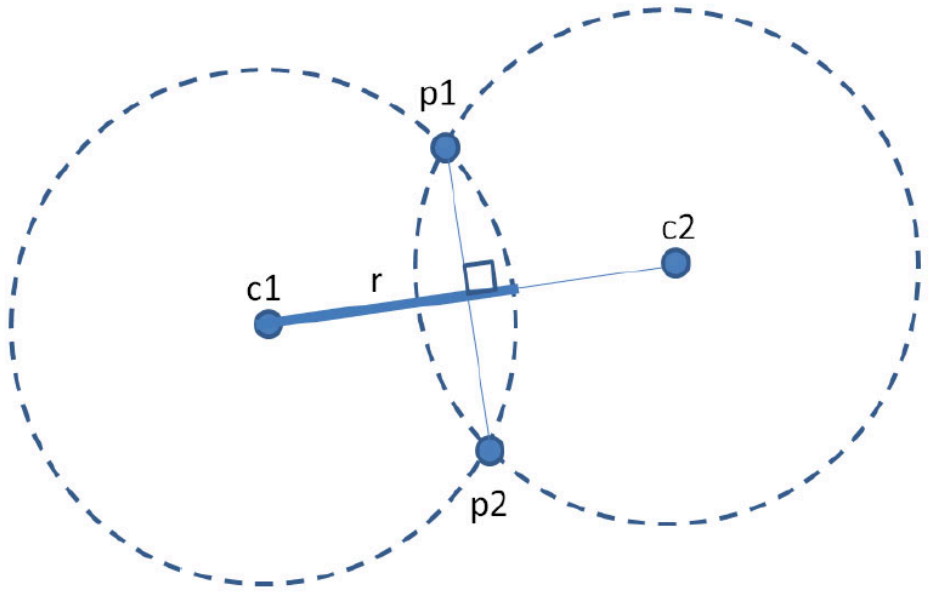


Figure 7.4: Circle Through 2 Points and Radius

```
boolean circle2PtsRad (point p1, point p2,
                                  double r, point c) {
  double d2 = (p1.x - p2.x)*(p1.x - p2.x)
            + (p1.y - p2.y)*(p1.y - p2.y);
  double det = r * r / d2 - 0.25;

  if (det < 0.0)
    return false;

  double h = Math.sqrt(det);

  c.x = (p1.x + p2.x) * 0.5
        + (p1.y - p2.y) * h;

  c.y = (p1.y + p2.y) * 0.5
        + (p2.x - p1.x) * h;

  return true;
}
```

Triangles

ch7_03_triangles.cpp/java

Triangles

- Polygon 3 vertices 3 edges
- Area of Triangle
  - A = 0.5 * base x height
- Perimeter p = a + b + c
  - a, b, c 3 edges

- Heron's formula
  - Area = sqrt(s*(s-a)*(s-b)*(s-c))
  - s = 0.5 * p : semi-perimeter
  - safer from overflow:
    - A = sqrt(s)*sqrt(s-a)*sqrt(s-b)*sqrt(s-c)
    - slightly more imprecise

```
double perimeter (double ab, double bc, double
ca) {
  return ab + bc + ca;
}

double perimeter (point a, point b, point c){
  return dist(a, b)
         + dist(b, c) + dist(c, a);
}

double area (double ab, double bc, double ca) {
  double s = 0.5 * perimeter(ab, bc, ca);
  return Math.sqrt(s)
         * Math.sqrt(s - ab)
         * Math.sqrt(s - bc)
         * Math.sqrt(s - ca);
}

double area (point a, point b, point c) {
  return area(dist(a, b),
              dist(b, c), dist(c, a));
}
```

Types of triangles

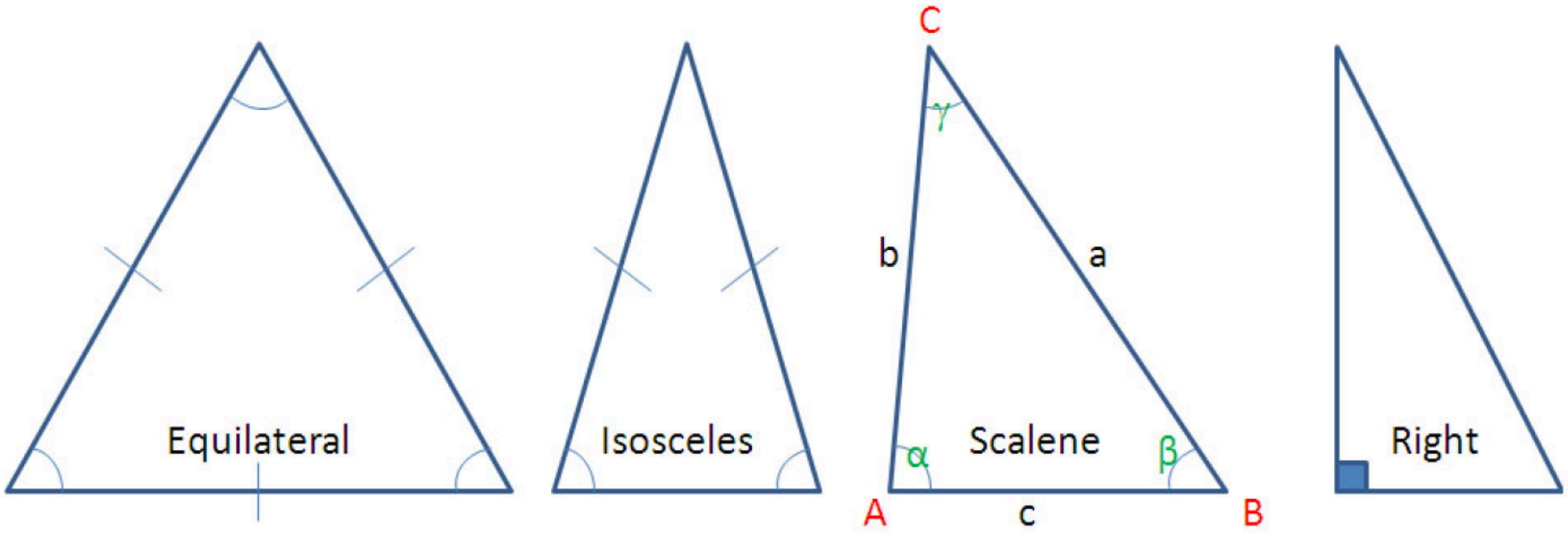
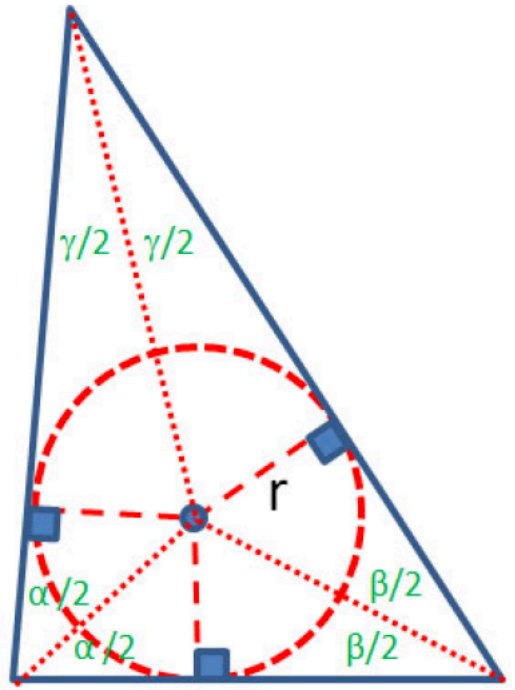


Figure 7.5: Triangles

InCircle

- 3 points p, q, r
- ? circumcenter c1 & radius R1
- of inner / inscribed circle / incircle



InCircle

- 3 points p, q, r
- ? circumcenter c1 & radius R1
- of inner / inscribed circle / incircle

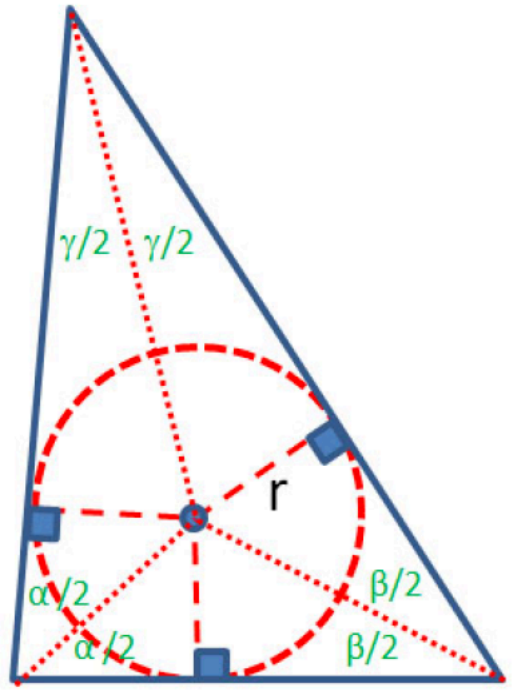
- triangle with area A and semi-perimeter s
  - inscribed circle (incircle) with
  - radius r = A/s

```
double rInCircle (double ab,
                  double bc, double ca) {
  return area(ab, bc, ca) / (0.5 *
perimeter(ab, bc, ca));
}

double rInCircle (point a, point b, point c) {
  return rInCircle( dist(a, b),
                    dist(b, c), dist(c,a));
}
```

InCircle

- 3 points p, q, r
- ? circumcenter c1 & radius R1
- of inner / inscribed circle / incircle

- center of incircle
  - meeting point between triangle's angle bisectors
  - 2 angle bisectors and find intersection



InCircle

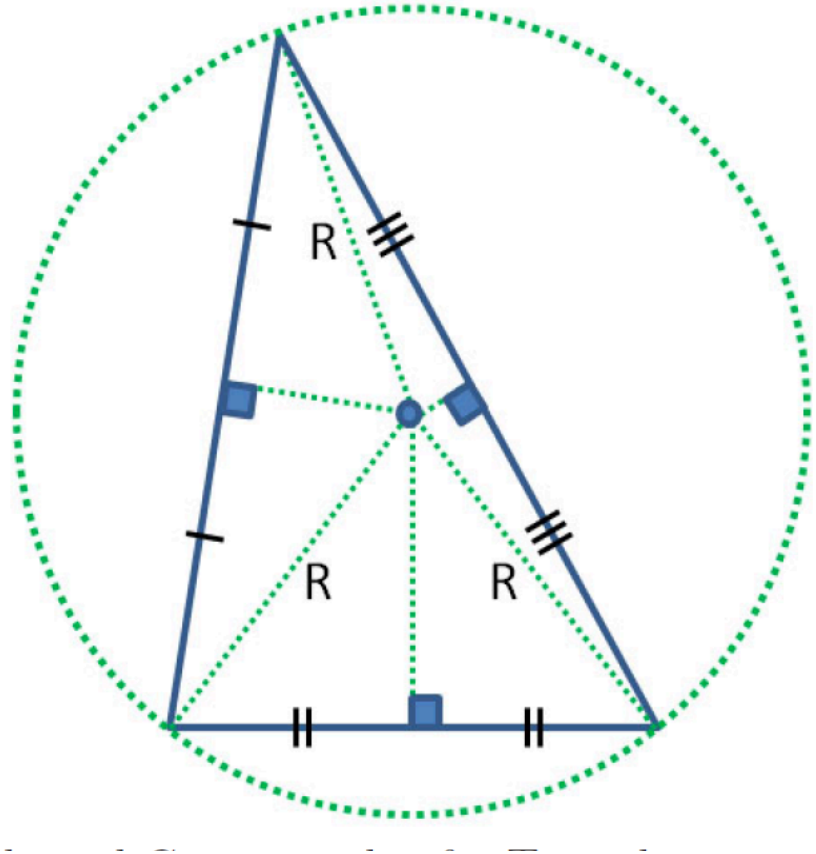
- 3 points p, q, r
- ? circumcenter c1 & radius R1
- of inner / inscribed circle / incircle

- returns
  - 1 if there is an inCircle
    - center ctr radius r
  - 0 otherwise

```
int inCircle (point p1, point p2, point p3,
                          point ctr, double r) {
  r = rInCircle(p1, p2, p3);
  if (Math.abs(r) < EPS)
    return 0;
  line l1 = new line(),
       l2 = new line();
  double ratio = dist(p1, p2) / dist(p1, p3);
  point p;
  p = translate(p2,
            scale(toVec(p2, p3),
                  ratio / (1 + ratio)));
  pointsToLine(p1, p, l1);
  ratio = dist(p2, p1) / dist(p2, p3);
  p = translate(p1,
            scale(toVec(p1, p3),
                  ratio / (1 + ratio)));
  pointsToLine(p2, p, l2);
  areIntersect(l1, l2, ctr);
  return 1;
}
```

CircumCircle

- 3 points p, q, r
- ? circumcenter c2 & radius R2
- of outer / circumscribed circle / circumcircle



CircumCircle

- 3 points p, q, r
- ? circumcenter c2 & radius R2
- of outer / circumscribed circle / circumcircle

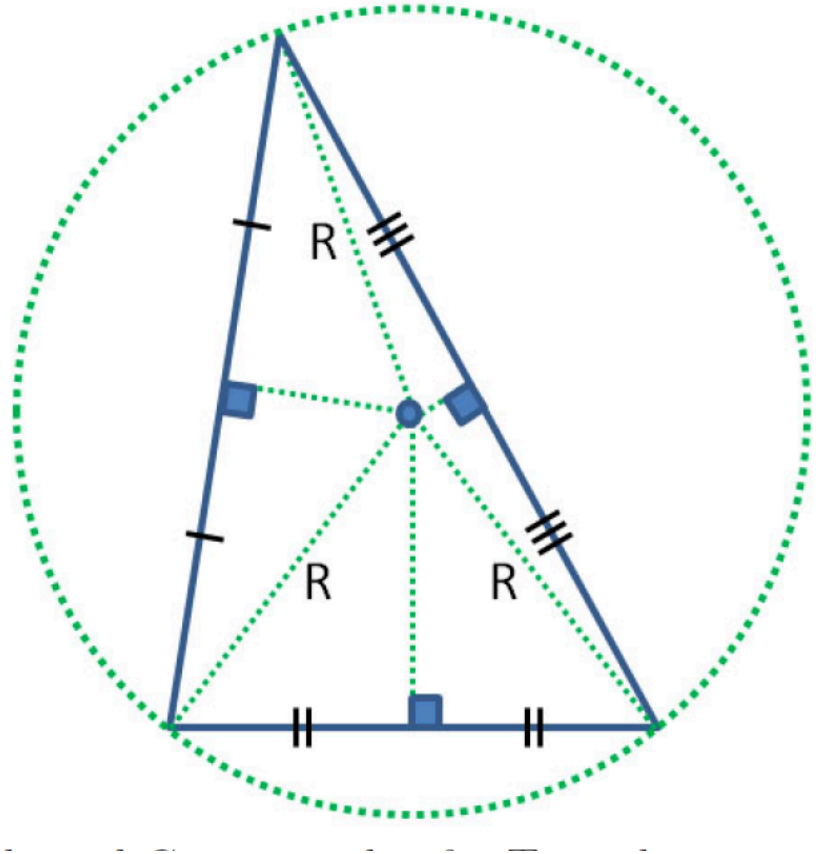
- 3 sides: a, b, c and area A
  - circumcircle radius
  - R = abc/(4A)

```
double rCircumCircle(double ab, double bc,
                     double ca) {
  return ab * bc * ca
        /(4.0 * area(ab, bc, ca));
}

double rCircumCircle(point a, point b,
                     point c) {
  return
    rCircumCircle(dist(a,b),
                  dist(b,c), dist(c,a));
}
```

CircumCircle

- 3 points p, q, r
- ? circumcenter c2 & radius R2
- of outer / circumscribed circle / circumcircle

- center = meeting point between triangle's perpendicular bisectors



CircumCircle

- 3 points p, q, r
- ? circumcenter c2 & radius R2
- of outer / circumscribed circle / circumcircle

```
double circumCircle(point p1, point p2,
                    point p3, point ctr) {
  double a = p2.x - p1.x,
         b = p2.y - p1.y;
  double c = p3.x - p1.x,
         d = p3.y - p1.y;
  double e = a * (p1.x + p2.x)
           + b * (p1.y + p2.y);
  double f = c * (p1.x + p3.x)
           + d * (p1.y + p3.y);
  double g = 2.0 * (a * (p3.y - p2.y)
           - b * (p3.x - p2.x));
  if (Math.abs(g) < EPS)
    return 0;

  ctr.x = (d * e - b * f) / g;
  ctr.y = (a * f - c * e) / g;

  return dist(p1, ctr);
}
```

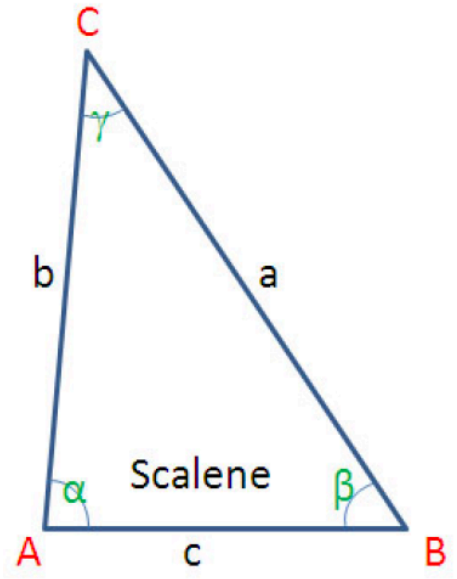
Triangle inequalities

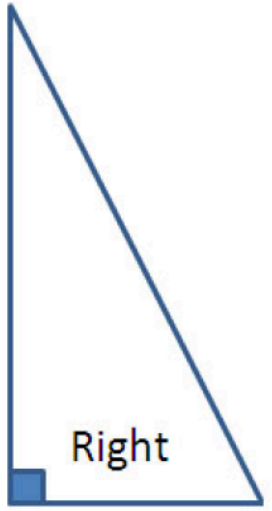
- To check if three line segments
  - length a <= b <= c
  - can form a triangle

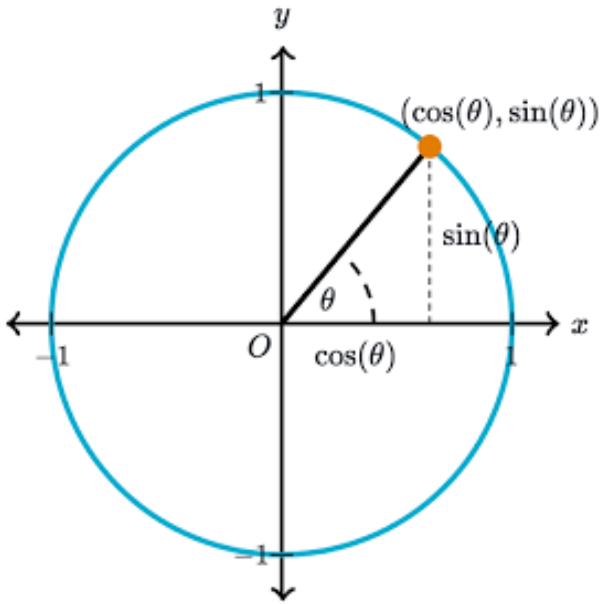
- check if a + b > c

Trigonometry

- Law of Cosines
  - $c^2 = a^2 + b^2 - 2ab\cos\gamma$
- Law of Sines
  - $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$
  - R radius of circumcircle
- Phytagorean Theorem
  - right triangle
  - $c^2 = a^2 + b^2$
    - $\cos\frac{\pi}{2} = 0$
- Pythagorean Triple (a, b, c)
  - such that $c^2 = a^2 + b^2$
  - if (a, b, c) is a Pythagorean triple
  - then so is (ka, kb, kc)
  - for any positive integer k

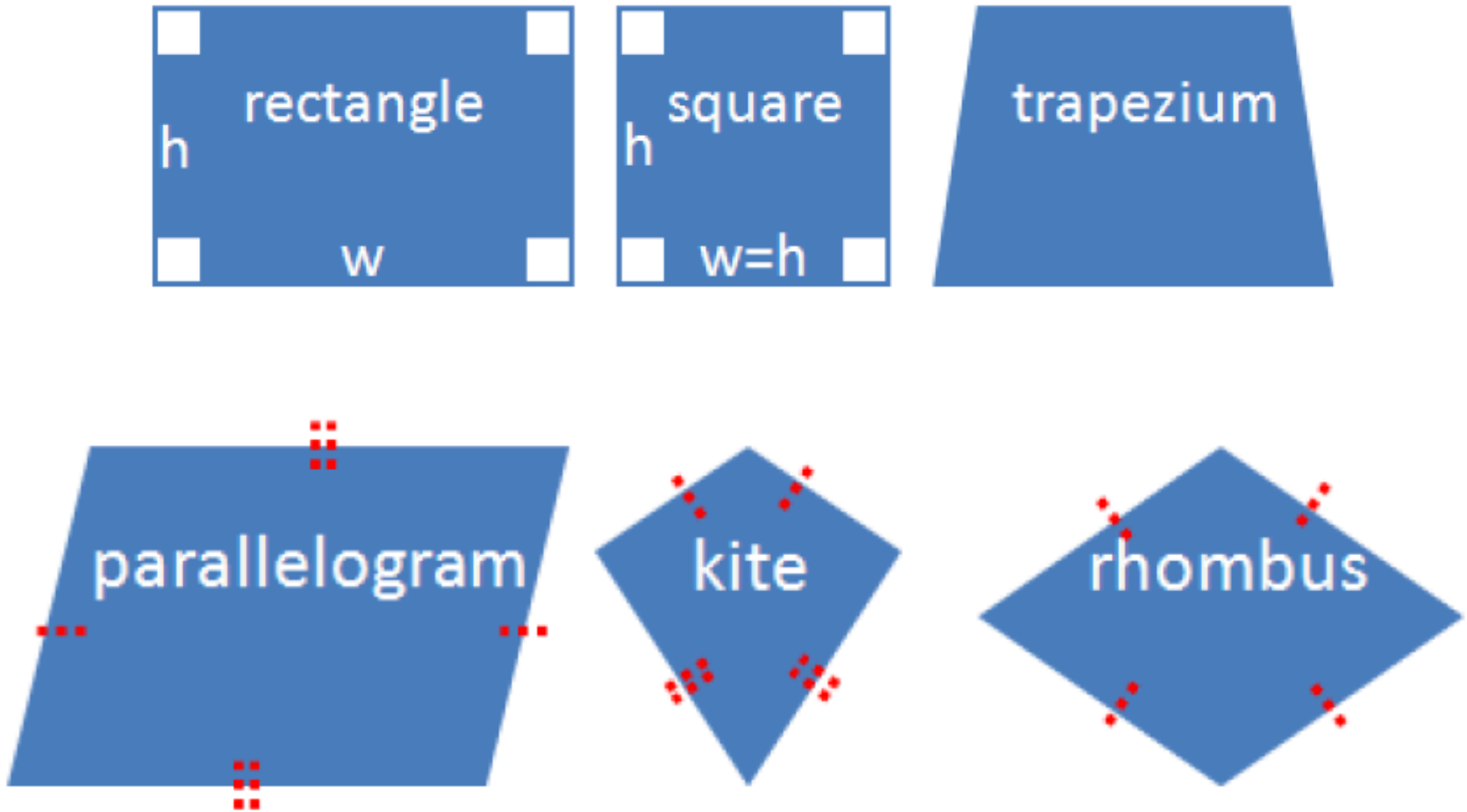






Dr Siba Hadar & Dr Antoun Yaacoub

Quadrilaterals

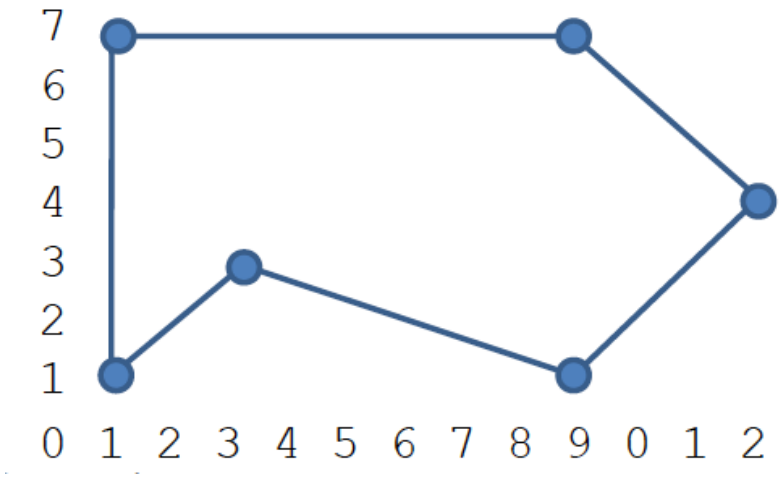


Algorithms on Polygon

ch7_04_polygon.cpp/java

Polygon

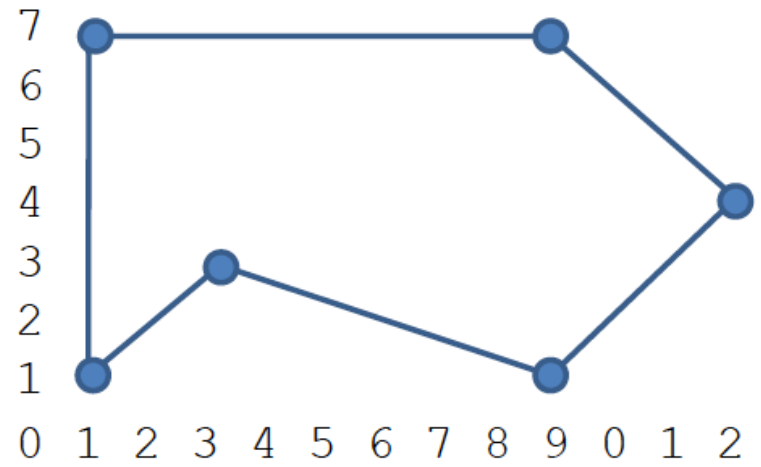
- plane figure bounded by
  - closed circuit
  - composed of finite sequence of straight line segments
- basic form
  - vertices ordered in cw or ccw order
- usually first = last vertex
- https://visualgo.net/en/polygon




```
List<point> P = new ArrayList<point>();
P.add(new point(1, 1));
P.add(new point(3, 3));
P.add(new point(9, 1));
P.add(new point(12, 4));
P.add(new point(9, 7));
P.add(new point(1, 7));
P.add(P.get(0)); // loop back
```

Perimeter of polygon (trivial)

- perimeter of polygon
  - convex or concave
  - with n vertices
  - given in CW or CCW
- computed via simple function




```
double perimeter(List<point> P) {

  double result = 0.0;
  for (int i = 0; i<(int)P.size()-1; i++)
    result += dist(P.get(i),P.get(i+1));

  return result;
}
```

Area of a Polygon

- simple polygon non-self-intersecting
  - $A = \frac{1}{2}\begin{bmatrix} x_1 & \cdots & y_1 \\ \vdots & \ddots & \vdots \\ x_n & \cdots & y_n \end{bmatrix}$
  - $= \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i)$
  - beware with last point
- A
  - >0 vertices CCW
  - <0 vertices CW
- https://en.wikipedia.org/wiki/Polygon#Area

```
double area(List<point> P) {
  double result = 0.0, x1, y1, x2, y2;
  for (int i = 0;
       i < (int)P.size()-1; i++) {
    x1 = P.get(i).x;
    x2 = P.get(i+1).x;
    y1 = P.get(i).y;
    y2 = P.get(i+1).y;
    result += (x1 * y2 - x2 * y1);
  }
  return Math.abs(result) / 2.0;
}
```

Check if a Polygon is Convex

- convex polygon
  - → any line segment drawn inside the polygon does not intersect any edge of the polygon
  - otherwise Concave

- convex?
  - check whether all 3 consecutive vertices form same turns
    - all left turns ccw, or
    - all right turns cw
  - 1 counter example? → concave

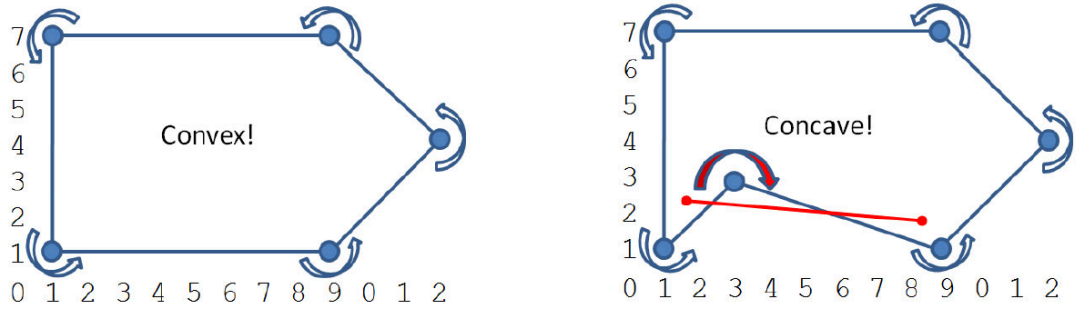
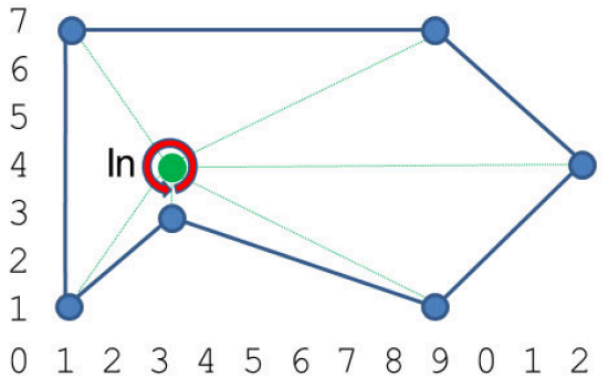


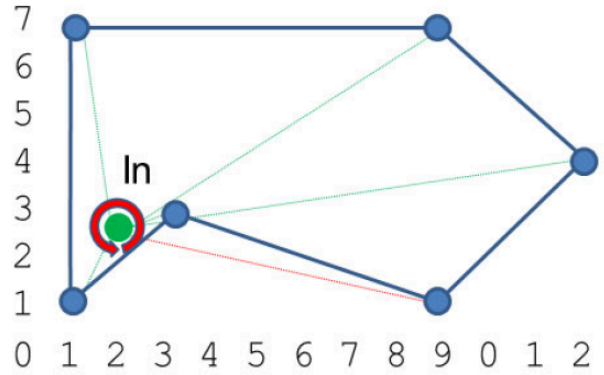
Figure 7.8: Left: Convex Polygon, Right: Concave Polygon

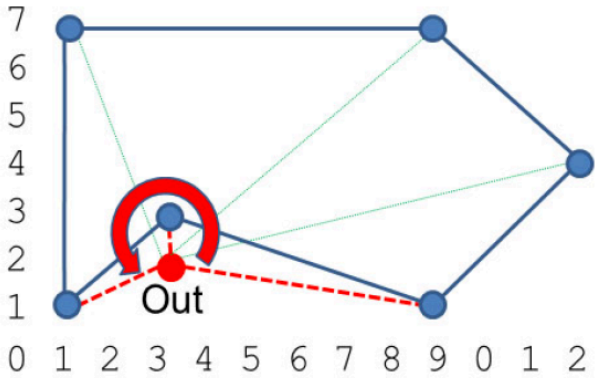
```
boolean isConvex(List<point> P) {
  int sz = (int)P.size();
  if (sz <= 3) return false;
  boolean isLeft =
    ccw(P.get(0), P.get(1), P.get(2));
  // remember one result
  for (int i = 1; i < sz-1; i++)
    if (ccw(P.get(i), P.get(i+1),
        P.get((i+2) == sz ? 1 : i+2))
      != isLeft)
      return false;
  return true;
}
```

Check if a Point is Inside a Polygon

- winding number algorithm
- check for either convex or concave polygon
- sum of angles $\Sigma p_i \widehat{ptp_{i+1}}$
  - CCW angle >0
  - CW angle <0
- if sum = 2π
  - pt is inside polygon P
- else→ outside







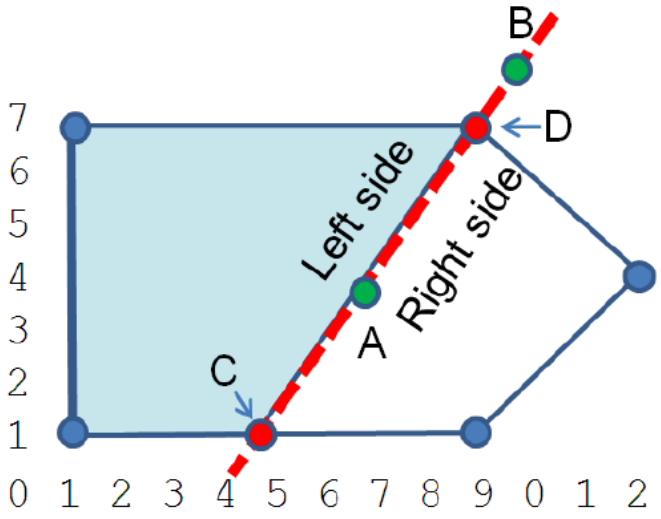
Check if a Point is Inside a Polygon

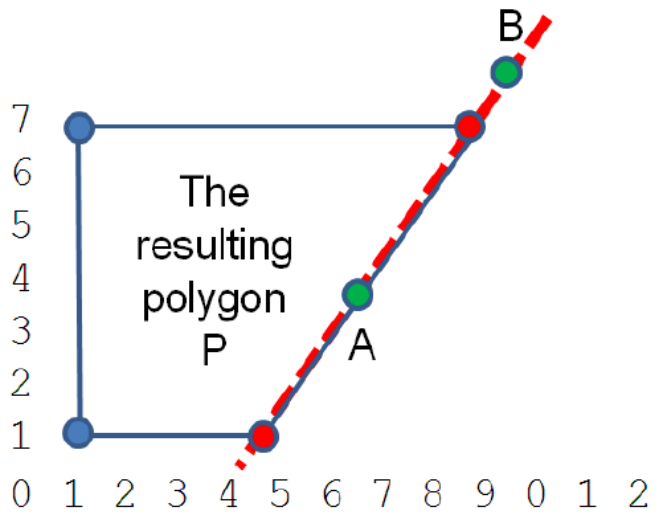
- winding number algorithm
- check for either convex or concave polygon
- sum of angles $\Sigma p_i \widehat{ptp_{i+1}}$
  - CCW angle >0
  - CW angle <0
- if sum = 2π
  - pt is inside polygon P
- else→ outside

```
boolean inPolygon(point pt, List<point> P) {
  if ((int)P.size() == 0)
    return false;
  double sum = 0;

  for (int i = 0; i < (int)P.size()-1; i++)
  {
    if (ccw(pt, P.get(i), P.get(i+1)))
      sum += angle(P.get(i), pt, P.get(i+1));
    else
      sum -= angle(P.get(i), pt, P.get(i+1));
  }

  return Math.abs(Math.abs(sum)-2*Math.PI)<EPS;

}
```

Cut a Polygon with a Straight Line

- line (AB) cut convex polygon Q
  - → 2 convex sub-polygons
- visit vertices v_i one by one
  - ABv_i CCW
    - → add v_i to P
  - $(AB) \cap [v_i v_{i+1}] = I \neq \emptyset$
    - → add I as new vertex in P if not a vertex already
  - ABv_i CW
    - → do nothing





Cut a Polygon with a Straight Line

- line (AB) cut convex polygon Q
  - → 2 convex sub-polygons
- visit vertices v_i one by one
  - ABv_i CCW
    - → add v_i to P
  - $(AB) \cap [v_i v_{i+1}] = I \neq \emptyset$
    - → add I as new vertex in P if not a vertex already
  - ABv_i CW
    - → do nothing
- can use areIntersect

```
point lineIntersectSeg(point p, point q,
                       point A, point B) {

  double a = B.y-A.y;
  double b = A.x-B.x;

  double c = B.x*A.y-A.x*B.y;

  double u = Math.abs(a*p.x+b*p.y+c);
  double v = Math.abs(a*q.x+b*q.y+c);

  return new point((p.x*v+q.x*u)/(u+v),
                   (p.y*v+q.y*u)/(u+v));
}
```

Cut a Polygon with a Straight Line

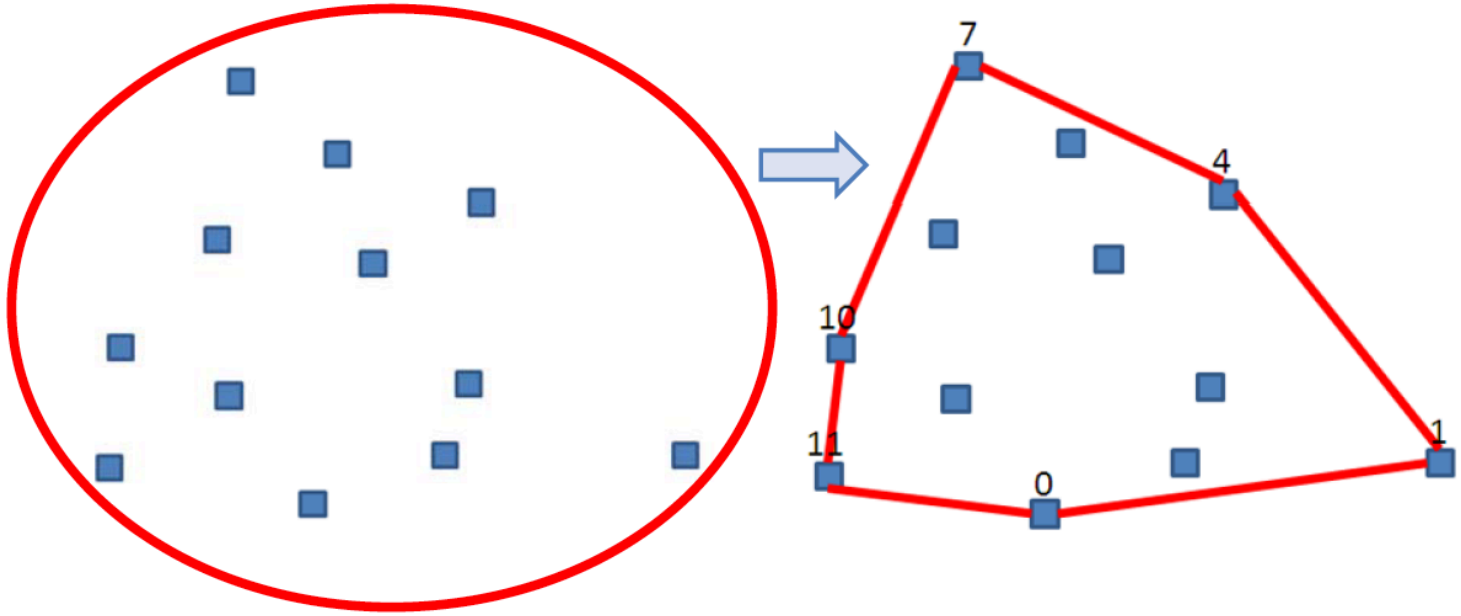
- line (AB) cut convex polygon Q
  - → 2 convex sub-polygons
- visit vertices v_i one by one
  - ABv_i CCW
    - → add v_i to P
  - $(AB) \cap [v_i v_{i+1}] = I \neq \emptyset$
    - → add I as new vertex in P if not a vertex already
  - ABv_i CW
    - → do nothing
- cross product slide

```
List<point> cutPolygon(point a, point b,
                       List<point> Q) {
  List<point> P = new ArrayList<point>();

  for (int i = 0; i < (int)Q.size(); i++) {
    double left1 = cross(toVec(a, b),
                         toVec(a, Q.get(i))),
           left2 = 0;

    if (i != (int)Q.size()-1)
      left2 = cross(toVec(a, b),
                    toVec(a, Q.get(i+1)));

    if (left1 > -EPS)
      P.add(Q.get(i));

    if (left1 * left2 < -EPS)
      P.add(lineIntersectSeg(Q.get(i),
                             Q.get(i+1),a,b));
  }

  if (!P.isEmpty()
      && P.get(P.size()-1).compareTo(P.get(0))!=0)
    P.add(P.get(0));

  return P;
}
```

Find Convex Hull of a Set of Points

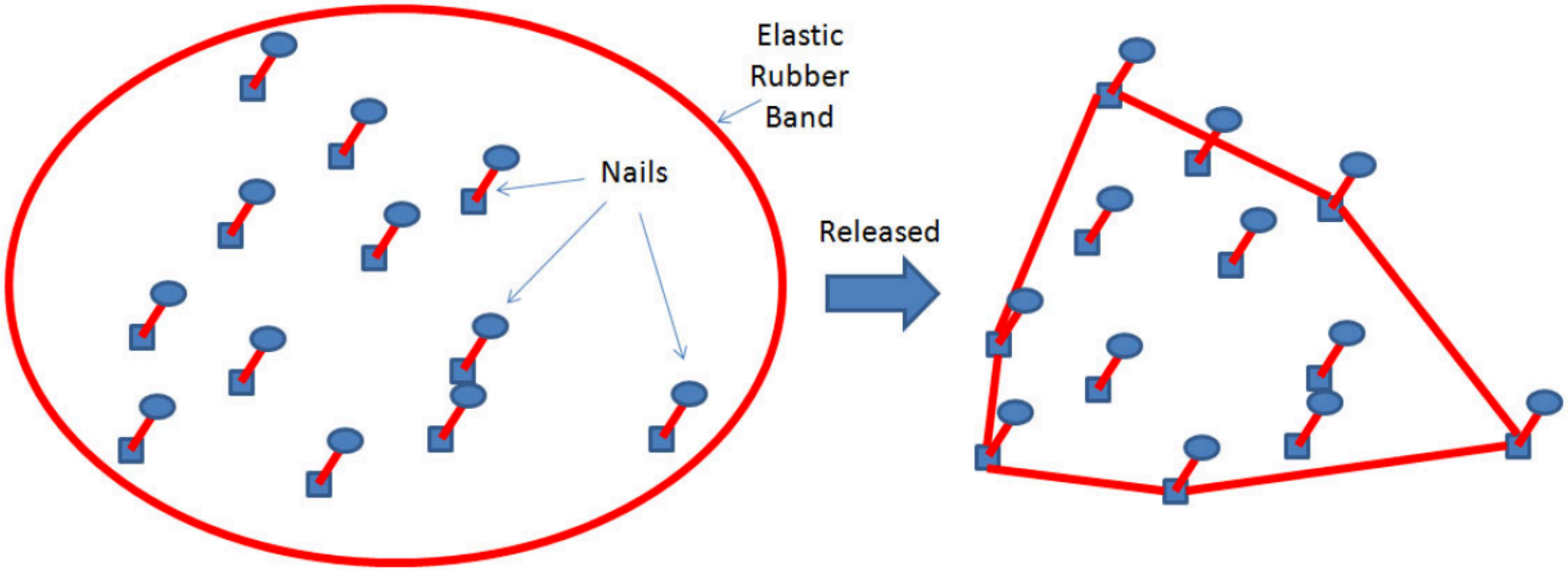
- The convex hull of set of points P
  - smallest convex polygon CH(P)
  - for which each point in P is
    - either on the boundary of CH(P) or
    - in its interior

- https://visualgo.net/en/convexhull



Find Convex Hull of a Set of Points

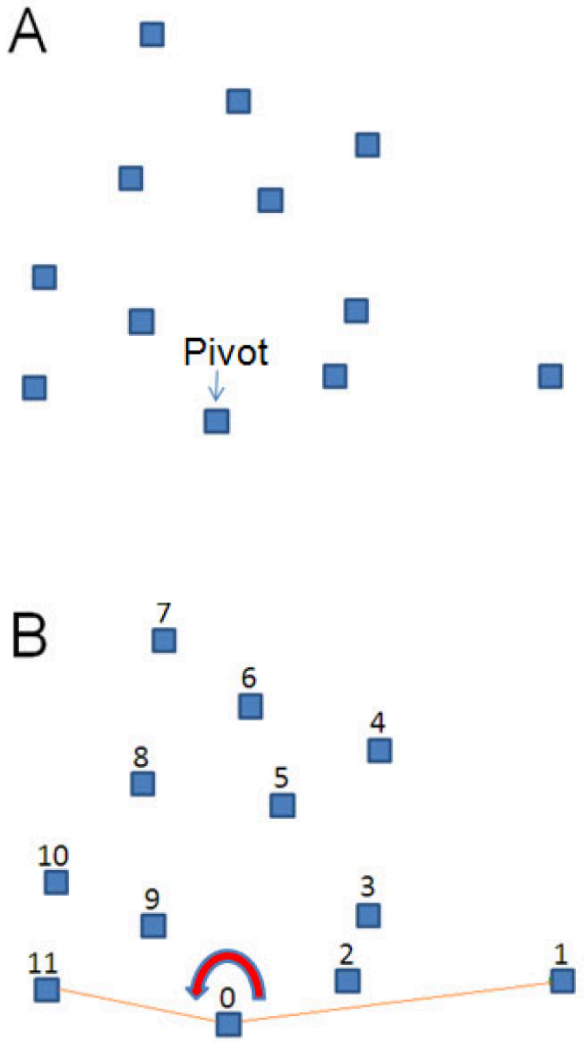
- if
  - points = nails
  - rubber band released → enclose as small an area as possible
- → convex hull



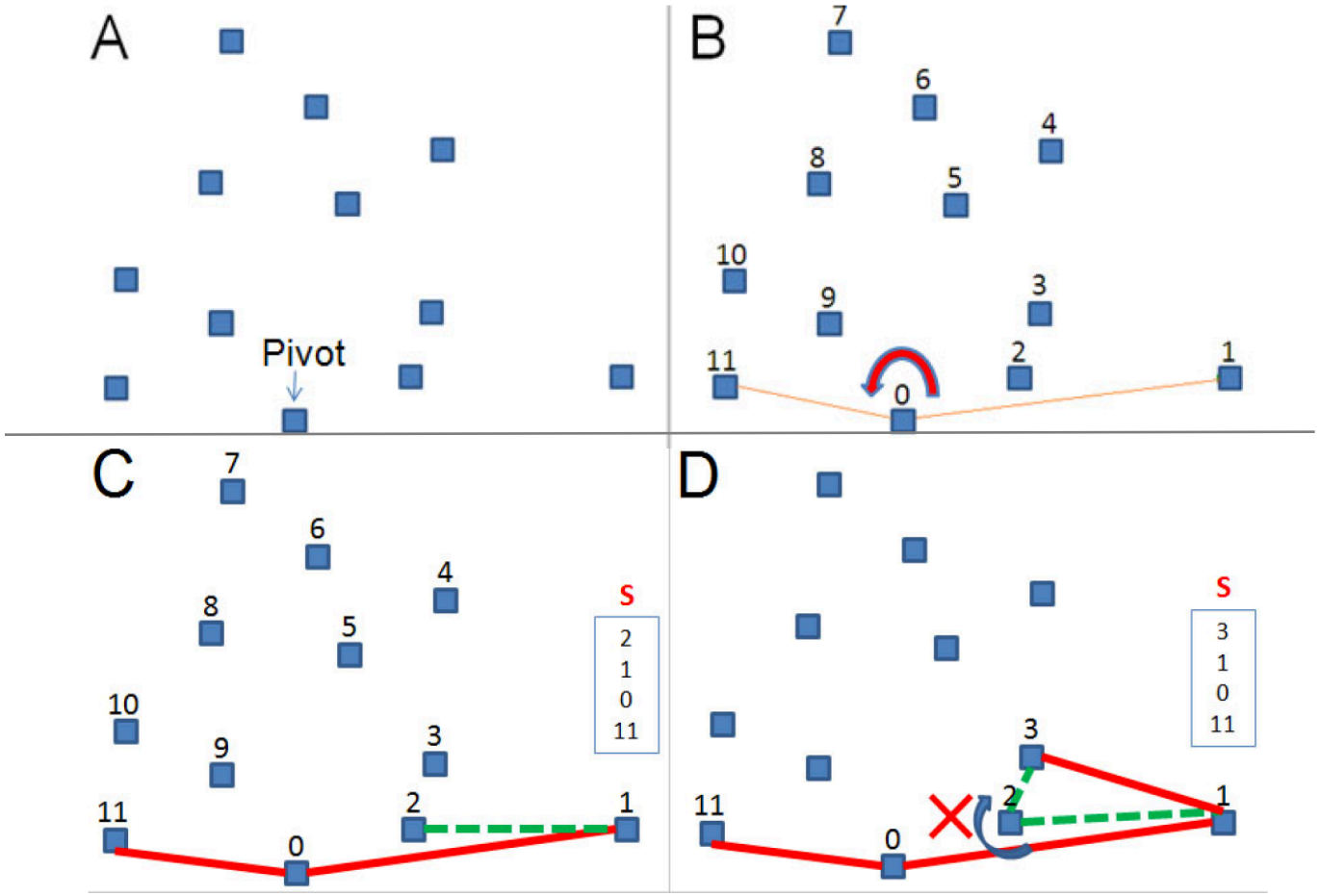
- Finding convex hull of a set of points has natural applications in packing problems.

Graham's Scan algorithm $O(n \log n)$

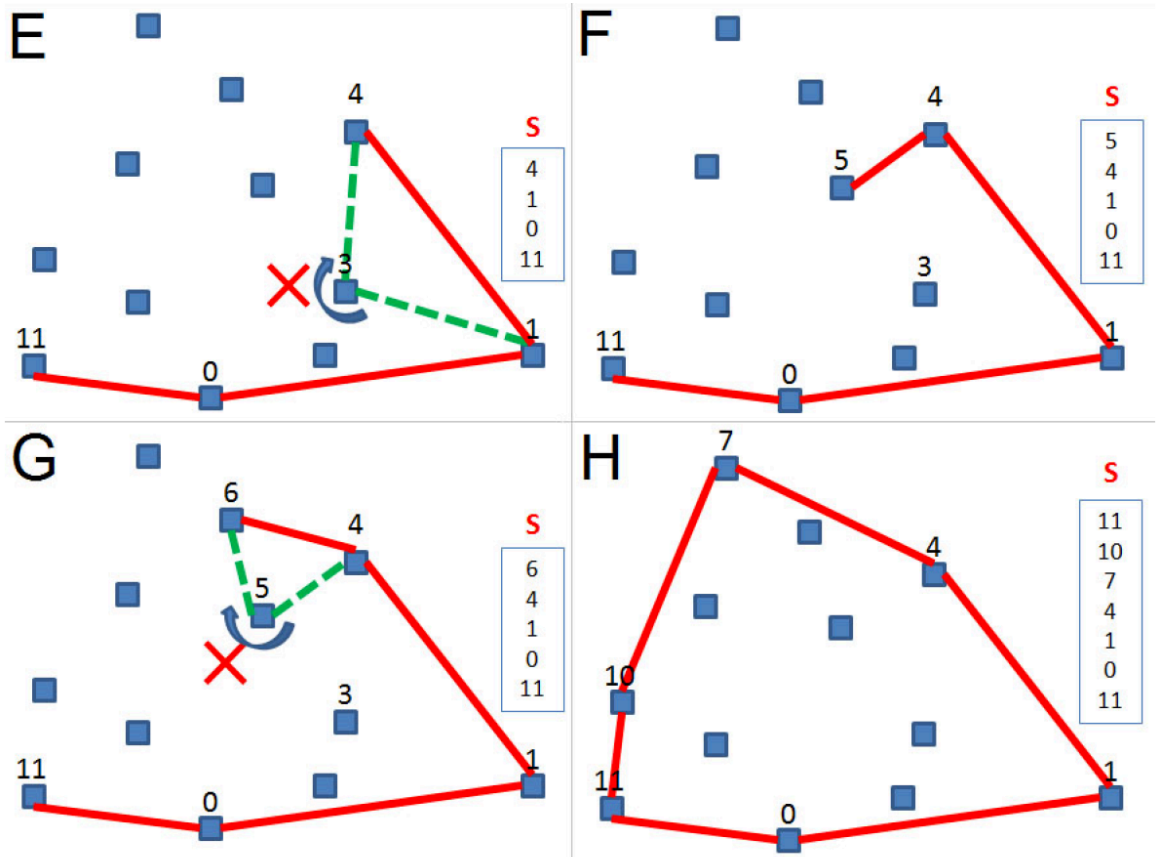
- pivot
  - bottom most, right most point
- angular sorting w.r.t pivot
  - easy with library
- series of ccw tests
  - stack S of candidate points
  - invariant
    - top 3 items in stack S must be CCW
  - initially
    - insert 3 points: N-1, 0, and 1
    - → always CCW
    - because 0 is the pivot!
  - then
    - each vertex v in sort
    - if top 2 with v are not going to be CCW
      - pop the last vertex added
    - else
      - push v int S



Graham's Scan algorithm



Graham's Scan algorithm



Graham's Scan algorithm

not the best programming practice

```
point pivot = new point();

List<point> CH(List<point> P) {
  int i, j, n = (int)P.size();
  if (n <= 3) {
    if (P.get(0).compareTo(P.get(n-1)) != 0)
      P.add(P.get(0));
    return P;
  }

  int P0 = 0;
  for (i = 1; i < n; i++)
    if (P.get(i).y  < P.get(P0).y
       ||(P.get(i).y == P.get(P0).y && P.get(i).x>P.get(P0).x))
      P0 = i;

  point temp = P.get(0);
  P.set(0, P.get(P0));
  P.set(P0 ,temp);
  pivot = P.get(0);
  Collections.sort(P, new AngComp());

  List<point> S = new ArrayList<point>();
  S.add(P.get(n-1));
  S.add(P.get(0));
  S.add(P.get(1));

  i = 2;
  while (i < n) {
    j = S.size() - 1;
    if (ccw(S.get(j-1), S.get(j), P.get(i)))
      S.add(P.get(i++));
    else S.remove(S.size() - 1);
  }
  return S;
}
```

```
class AngCom implements Comparator<point>{
  public int compare(point a, point b) {
    if (collinear(pivot, a, b))
      return dist(pivot, a) < dist(pivot, b) ? -1 : 1;
    double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (Math.atan2(d1y,d1x)-Math.atan2(d2y,d2x))<0?-1:1;
  }
}
```